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Anisotropic reflection coefficients for a weak-contrast interface

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SUMMARY

In this article the interaction of plane waves with a weak-contrast interface between two weakly anisotropic half-spaces is investigated. The anisotropy dealt with is of a general type. The stress–displacement vectors of the plane waves are calculated by perturbation theory. By assuming that the jump in elastic parameters and density across the interface is small, one can derive a simple expression for the R_{qPqP} coefficient. In cases in which the wave motion is restricted to a symmetry plane of an anisotropic medium, simple expressions for the R_{qSVqSV} and R_{SHSH} coefficients are also derived.

Key words: anisotropy, perturbation methods.

INTRODUCTION

Exact explicit formulae for plane-wave reflection coefficients can be obtained for isotropic (Červený, Molotkov & Pšenčík 1977) and transversely isotropic (Daley & Hron 1977) elastic media. For the general type of anisotropy only a numerical solution is possible (Fedorov 1968). In a recent paper we derived approximate formulae for two weakly anisotropic half-spaces in welded contact (Zillmer, Gajewski & Kashtan 1997, hereafter referred to as Paper I).

The expressions obtained can be simplified by introducing a further restriction on the model: the elasticity tensors and densities of both half-spaces differ only by a small degree. This concept of two similar half-spaces was previously applied to isotropic (Bortfeld 1961; Shuey 1985) and transversely isotropic media (Banik 1987; Thomsen 1993). The approximate solution shows which medium parameters are responsible for the reflection and transmission of the different types of waves. This makes it possible to obtain information about the elastic parameters from reflection amplitudes (amplitude variation with offset and with azimuth) by inversion schemes (Mallik 1993). The approximate coefficients were also used to investigate the scattering in vertically inhomogeneous isotropic media (Richards & Frasier 1976).

In this paper we derive a simple expression for the R_{qPqP} coefficient for two weakly anisotropic half-spaces of general anisotropy. We further derive simple expressions for the R_{qSVqSV} and R_{SHSH} coefficients for the case that the incident quasi-shear waves propagate in a vertical symmetry plane of an anisotropic half-space.

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THEORY

The stress–displacement vectors of the plane waves which propagate in a weakly anisotropic medium can be calculated by perturbation theory (Paper I). In the following we first summarize the calculation and we then demonstrate the simplification of the linear boundary conditions in the case of a weak-constrast interface.

The stress–displacement vectors are the eigenvectors $\mathbf{Z}$ of the non-symmetric real 6×6 matrix $\mathbf{A}$:

$$\mathbf{Z}^{-1}\mathbf{A}\mathbf{Z} = \mathbf{Q}, \tag{1}$$

$$\mathbf{A} = \begin{pmatrix} -p\mathbf{C}_{33}^{-1}\mathbf{C}_{31} & -\mathbf{C}_{33}^{-1} \\ p^2(\mathbf{C}_{11} - \mathbf{C}_{13}\mathbf{C}_{33}^{-1}\mathbf{C}_{31}) - \rho\mathbf{I} & -p\mathbf{C}_{13}\mathbf{C}_{33}^{-1} \end{pmatrix}. \tag{2}$$

Here p is the horizontal slowness, $(\mathbf{C}_{ik})_{jl} = c_{ijkl}$ is the elasticity tensor and ρ is the density. $\mathbf{Q}$ is the diagonal matrix of eigenvalues, which are the vertical slownesses of the plane waves. The eigenvectors $\mathbf{Z}$ can be calculated by perturbing the eigenvectors $\mathbf{Z}^{(0)}$ of an isotropic matrix $\mathbf{A}^{(0)}$, that is

$$\mathbf{A} \approx \mathbf{A}^{(0)} + \varepsilon\mathbf{A}^{(1)}, \tag{3}$$

$$\mathbf{Z} \approx \mathbf{Z}^{(0)} + \varepsilon\mathbf{Z}^{(1)} = \mathbf{X}^{(0)}\mathbf{F} + \varepsilon\mathbf{X}^{(0)}\mathbf{F}\mathbf{H}. \tag{4}$$

The indices (0) and (1) denote the order in the small parameter ε. The reference eigenvalue problem for the matrix $\mathbf{A}^{(0)}$ can be solved explicitly (Paper I, eqs A1–A6). The matrix $\mathbf{X}^{(0)}$ is the matrix of eigenvectors of $\mathbf{A}^{(0)}$. In zero-order perturbation theory we obtain linear combinations of those vectors of the basis $\mathbf{X}^{(0)}$ which belong to the same eigenvalue (degenerate perturbation theory, Landau & Lifshitz 1958). These eigenvectors form a basis $\mathbf{Z}^{(0)} = \mathbf{X}^{(0)}\mathbf{F}$. The first-order perturbation $\mathbf{Z}^{(1)}$ to the eigenvectors $\mathbf{Z}^{(0)}$ is calculated as a linear combination of the basis vectors $\mathbf{Z}^{(0)}$. The elements of $\mathbf{F}$ (eq. A2) and $\mathbf{H}$ (eq. A4) are given in terms of the elements of the matrices

S and **V**, defined as

$$\mathbf{S}=\mathbf{X}^{(0)-1}\mathbf{A}^{(1)}\mathbf{X}^{(0)}, \tag{5}$$

$$\mathbf{V}=\mathbf{Z}^{(0)-1}\mathbf{A}^{(1)}\mathbf{Z}^{(0)}. \tag{6}$$

The eigenvalue problem (1) has to be solved for both half-spaces. In the case of a boundary with a weak contrast in elastic moduli and in density we use the same isotropic reference medium for both half-spaces. We then introduce perturbations to the elasticity tensor and density:

$$c_{ijkl}^{[1]}=c_{ijkl}^{(0)}+\varepsilon c_{ijkl}^{(1)[1]}, \qquad c_{ijkl}^{[2]}=c_{ijkl}^{(0)}+\varepsilon c_{ijkl}^{(1)[2]}, \tag{7}$$

$$c_{ijkl}^{(0)}=\lambda^{(0)}\delta_{ij}\delta_{kl}+\mu^{(0)}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{8}$$

$$\rho^{[1]}=\rho^{(0)}+\varepsilon\rho^{(1)[1]}, \qquad \rho^{[2]}=\rho^{(0)}+\varepsilon\rho^{(1)[2]}. \tag{9}$$

The indices [1] and [2] denote the two half-spaces.

We have the same matrix $\mathbf{A}^{(0)}$ for both half-spaces and therefore the same solution for the eigenvectors:

$$\mathbf{X}^{(0)}=\mathbf{X}^{(0)[1]}=\mathbf{X}^{(0)[2]}. \tag{10}$$

The difference between this and the perturbation theory presented in Paper I is that we now have additionally a perturbation in density. As a result, the matrices $\mathbf{A}^{(1)}$ and $\mathbf{S}$ contain terms proportional to $\rho^{(1)}$. The matrix $\mathbf{A}^{(1)}$ is given in Paper I, eqs (16), (17) and (A7). The following elements have changed:

$$A_{41}^{(1)}=p^2\left(c_{11}^{(1)}-\frac{2\lambda^{(0)}}{\lambda^{(0)}+2\mu^{(0)}}c_{13}^{(1)}+\frac{\lambda^{(0)2}}{(\lambda^{(0)}+2\mu^{(0)})^2}c_{33}^{(1)}\right)-\rho^{(1)},$$

$$A_{52}^{(1)}=p^2c_{66}^{(1)}-\rho^{(1)}, \qquad A_{63}^{(1)}=-\rho^{(1)}. \tag{11}$$

The matrix $\mathbf{S}$ is given in Appendix A (eq. A1).

The continuity of the displacement vector and the stress vector at the interface leads to a system of linear equations. For the isotropic reference case we write

$$\mathbf{N}^{(0)}\mathbf{r}_{\mathrm{inc}}=\mathbf{x}_{\mathrm{inc}}^{(0)}. \tag{12}$$

The index inc refers to the incident wave.

The coefficient matrix $\mathbf{N}^{(0)}$ contains the stress–displacement vectors of the three waves of each half-space which leave the interface. Here we have two equal half-spaces and therefore

$$\mathbf{N}^{(0)}=\mathbf{X}^{(0)}\,\mathrm{diag}(-1,\,-1,\,-1,1,1,1). \tag{13}$$

In the case of an incident *P* wave the solution vector is given by

$$\mathbf{r}_P=(R_{PP},R_{PSH},R_{PSV},T_{PP},T_{PSH},T_{PSV})^{\mathrm{T}}$$
$$=(0,0,0,1,0,0)^{\mathrm{T}}. \tag{14}$$

The solution in zero-order perturbation theory for two weakly anisotropic half-spaces can be calculated using the following equation (Paper I, eqs 36–39):

$$\mathbf{r}_{\mathrm{inc}}^{(0)}=\mathbf{G}^{\mathrm{T}}\mathbf{r}_{\mathrm{inc}}, \tag{15}$$

with the matrix $\mathbf{G}$ given by

$$\mathbf{G}=\mathbf{F}^{[1]}\,\mathrm{diag}(1,1,1,0,0,0)+\mathbf{F}^{[2]}\,\mathrm{diag}(0,0,0,1,1,1). \tag{16}$$

In the case of an incident *qP* wave we obtain the following:

$$\mathbf{r}_{qP}^{(0)}=(R_{qPqP}^{(0)},R_{qPqS1}^{(0)},R_{qPqS2}^{(0)},T_{qPqP}^{(0)},T_{qPqS1}^{(0)},T_{qPqS2}^{(0)})^{\mathrm{T}}$$
$$=\mathbf{G}^{\mathrm{T}}\mathbf{r}_P=(0,0,0,1,0,0)^{\mathrm{T}}. \tag{17}$$

To first order we have (Paper I, eqs 40–43)

$$\mathbf{r}_{\mathrm{inc}}^{(1)}=\mathbf{G}^{\mathrm{T}}\mathbf{N}^{(0)-1}(\mathbf{z}_{\mathrm{inc}}^{(1)[1]}-\mathbf{N}^{(1)}\mathbf{r}_{\mathrm{inc}}^{(0)}), \tag{18}$$

with the matrix $\mathbf{N}^{(1)}$ given by

$$\mathbf{N}^{(1)}=\mathbf{X}^{(0)}\mathbf{F}^{[1]}\mathbf{H}^{[1]}\,\mathrm{diag}(-1,\,-1,\,-1,0,0,0)$$
$$+\mathbf{X}^{(0)}\mathbf{F}^{[2]}\mathbf{H}^{[2]}\,\mathrm{diag}(0,0,0,1,1,1). \tag{19}$$

In the case of an incident *qP* wave we obtain the following:

$$\mathbf{r}_{qP}^{(1)}=\mathbf{G}^{\mathrm{T}}\mathbf{N}^{(0)-1}(\mathbf{z}_4^{(1)[1]}-\mathbf{z}_4^{(1)[2]}) \tag{20}$$
$$=\mathbf{G}^{\mathrm{T}}\,\mathrm{diag}(-1,\,-1,\,-1,1,1,1)(\mathbf{F}^{[1]}\mathbf{H}^{[1]}-\mathbf{F}^{[2]}\mathbf{H}^{[2]})$$
$$\times(0,0,0,1,0,0)^{\mathrm{T}}.$$

The first element of the vector (20) yields

$$R_{qPqP}=R_{qPqP}^{(0)}+\varepsilon R_{qPqP}^{(1)} \tag{21}$$
$$=\varepsilon\Delta h_{41}^{(1)}=-\varepsilon\frac{\Delta S_{14}}{2q_P}$$
$$=\varepsilon i\frac{1}{4q_P^2\rho^{(0)}}[p^4\Delta c_{11}+2p^2q_P^2\Delta c_{13}+q_P^4\Delta c_{33}-4p^2q_P^2\Delta c_{55}$$
$$+(q_P^2-p^2)\Delta\rho]$$
$$=\varepsilon i\left[\frac{1}{4}\left(\frac{\Delta c_{33}}{\lambda^{(0)}+2\mu^{(0)}}+\frac{\Delta\rho}{\rho^{(0)}}\right)-\left(\frac{p}{q_P}\right)^2\frac{1}{4}\frac{\Delta\rho}{\rho^{(0)}}\right.$$
$$\left.+p^2\frac{1}{4}\frac{2\Delta c_{13}-\Delta c_{33}-4\Delta c_{55}}{\rho^{(0)}}+p^2\left(\frac{p}{q_P}\right)^2\frac{1}{4}\frac{\Delta c_{11}}{\rho^{(0)}}\right].$$

Here Δ represents the difference in the quantities of both half-spaces, for example $\Delta h_{41}^{(1)}=h_{41}^{(1)[2]}-h_{41}^{(1)[1]}$.

Next we consider incident *qS* waves, which propagate in a vertical symmetry plane of half-space [1]. In this case the quasi-shear waves have *qSV* and *SH* polarizations. We then have $\mathbf{F}^{[1]}=\mathbf{I}$. Note that for the following derivation it is not necessary that the plane of incidence is a symmetry plane of half-space [2].

For an incident *SH* wave we obtain the following:

$$\mathbf{r}_{SH}=(R_{SHP},R_{SHSH},R_{SHSV},T_{SHP},T_{SHSH},T_{SHSV})^{\mathrm{T}}$$
$$=(0,0,0,0,1,0)^{\mathrm{T}} \tag{22}$$

and

$$\mathbf{r}_{SH}^{(0)}=(R_{SHqP}^{(0)},R_{SHSH}^{(0)},R_{SHqSV}^{(0)},T_{SHqP}^{(0)},T_{SHqS1}^{(0)},T_{SHqS2}^{(0)})^{\mathrm{T}}$$
$$=\mathbf{G}^{\mathrm{T}}\mathbf{r}_{SH}=(0,0,0,0,f_{55}^{(0)[2]},\,-f_{56}^{(0)[2]})^{\mathrm{T}}, \tag{23}$$

$$\mathbf{r}_{SH}^{(1)}=\mathbf{G}^{\mathrm{T}}\mathbf{N}^{(0)-1}(\mathbf{z}_5^{(1)[1]}-f_{55}^{(0)[2]}\mathbf{z}_5^{(1)[2]}+f_{56}^{(0)[2]}\mathbf{z}_6^{(1)[2]})$$
$$=\mathbf{G}^{\mathrm{T}}\,\mathrm{diag}(-1,\,-1,\,-1,1,1,1)[\mathbf{F}^{[1]}\mathbf{H}^{[1]}(0,0,0,0,1,0)^{\mathrm{T}}$$
$$-\mathbf{F}^{[2]}\mathbf{H}^{[2]}(0,0,0,0,f_{55}^{(0)[2]},\,-f_{56}^{(0)[2]})^{\mathrm{T}}]. \tag{24}$$

The second element of the vector (24) yields

$$R_{SHSH}=R_{SHSH}^{(0)}+\varepsilon R_{SHSH}^{(1)} \tag{25}$$
$$=\varepsilon[-h_{52}^{(1)[1]}+f_{55}^{(0)[2]}(f_{22}^{(0)[2]}h_{52}^{(1)[2]}-f_{23}^{(0)[2]}h_{53}^{(1)[2]})$$
$$-f_{56}^{(0)[2]}(f_{22}^{(0)[2]}h_{62}^{(1)[2]}-f_{23}^{(0)[2]}h_{63}^{(1)[2]})]$$
$$=-\varepsilon\frac{\Delta S_{25}}{2q_S}$$
$$=\varepsilon i\frac{1}{4q_S^2\mu^{(0)}}[-q_S^2\Delta c_{44}+p^2\Delta c_{66}-\Delta\rho]$$
$$=\varepsilon i\left[-\frac{1}{4}\left(\frac{\Delta c_{44}}{\mu^{(0)}}+\frac{\Delta\rho}{\rho^{(0)}}\right)+\left(\frac{p}{q_S}\right)^2\frac{1}{4}\left(\frac{\Delta c_{66}}{\mu^{(0)}}-\frac{\Delta\rho}{\rho^{(0)}}\right)\right].$$

Here we used eqs (A2)–(A4).

In the case of an incident *qSV* wave we obtain the following:

$$\mathbf{r}_{SV}=(R_{SVP},R_{SVSH},R_{SVSV},T_{SVP},T_{SVSH},T_{SVSV})^{\mathrm{T}}$$
$$=(0,0,0,0,0,1)^{\mathrm{T}} \tag{26}$$

and

$$\mathbf{r}^{(0)}_{qSV} = (R^{(0)}_{qSVqP}, R^{(0)}_{qSVSH}, R^{(0)}_{qSVqSV}, T^{(0)}_{qSVqP}, T^{(0)}_{qSVqS1}, T^{(0)}_{qSVqS2})^{\mathrm{T}}$$
$$= \mathbf{G}^{\mathrm{T}}\mathbf{r}_{SV} = (0, 0, 0, 0, f^{(0)[2]}_{56}, f^{(0)[2]}_{55})^{\mathrm{T}}, \quad (27)$$
$$\mathbf{r}^{(1)}_{qSV} = \mathbf{G}^{\mathrm{T}}\mathbf{N}^{(0)-1}(\mathbf{z}^{(1)[1]}_{6} - f^{(0)[2]}_{56}\mathbf{z}^{(1)[2]}_{5} - f^{(0)[2]}_{55}\mathbf{z}^{(1)[2]}_{6})$$
$$= \mathbf{G}^{\mathrm{T}}\,\mathrm{diag}(-1, -1, -1, 1, 1, 1)[\mathbf{F}^{[1]}\mathbf{H}^{[1]}(0, 0, 0, 0, 0, 1)^{\mathrm{T}} - \mathbf{F}^{[2]}\mathbf{H}^{[2]}(0, 0, 0, 0, f^{(0)[2]}_{56}, f^{(0)[2]}_{55})^{\mathrm{T}}]. \quad (28)$$

The third element of the vector (28) yields

$$R_{qSVqSV} = R^{(0)}_{qSVqSV} + \varepsilon R^{(1)}_{qSVqSV} \quad (29)$$
$$= \varepsilon[-h^{(1)[1]}_{63} + f^{(0)[2]}_{56}(f^{(0)[2]}_{23}h^{(1)[2]}_{52} + f^{(0)[2]}_{22}h^{(1)[2]}_{53}) + f^{(0)[2]}_{55}(f^{(0)[2]}_{23}h^{(1)[2]}_{62} + f^{(0)[2]}_{22}h^{(1)[2]}_{63})]$$
$$= -\varepsilon\frac{\Delta S_{36}}{2q_S}$$
$$= \varepsilon i\frac{1}{4q_S^2\rho^{(0)}}[p^2q_S^2\Delta c_{11} - 2p^2q_S^2\Delta c_{13} + p^2q_S^2\Delta c_{33} - (q_S^2-p^2)^2\Delta c_{55} - (q_S^2-p^2)\Delta\rho]$$
$$= \varepsilon i\left[-\frac{1}{4}\left(\frac{\Delta c_{55}}{\mu^{(0)}} + \frac{\Delta\rho}{\rho^{(0)}}\right) + \left(\frac{p}{q_S}\right)^2\frac{1}{4}\frac{\Delta\rho}{\rho^{(0)}} + p^2\frac{1}{4}\frac{\Delta c_{11} - 2\Delta c_{13} + \Delta c_{33} + 3\Delta c_{55}}{\rho^{(0)}} - p^2\left(\frac{p}{q_S}\right)^2\frac{1}{4}\frac{\Delta c_{55}}{\rho^{(0)}}\right].$$

The reflection coefficients are proportional to the imaginary unit. This is due to the normalization of the stress–displacement vectors (Paper I, eqs A3–A6). The vectors which correspond to the coefficients (21), (25) and (29) contain a factor $1/i$, so the factors i cancel in the physically relevant product of the reflection coefficient with the stress–displacement vector.

Finally, we introduce the phase angles ϑ_p and ϑ_s, defined with the help of $c^{(0)}_{ijkl}$ and $\rho^{(0)}$. We can identify these angles with the phase angles of the incident waves in half-space [1] because the differences $c^{[1]}_{ijkl} - c^{(0)}_{ijkl}$ and $\rho^{[1]} - \rho^{(0)}$ are of order ε, the reflection coefficients are also of order ε, and we neglect terms $\sim\varepsilon^2$. Our final result is

$$R_{qPqP} = \varepsilon i\left[\frac{1}{4}\left(\frac{\Delta c_{33}}{\lambda^{(0)} + 2\mu^{(0)}} + \frac{\Delta\rho}{\rho^{(0)}}\right) - \frac{1}{4}\frac{\Delta\rho}{\rho^{(0)}}\tan^2\vartheta_p + \frac{1}{4}\frac{2\Delta c_{13} - \Delta c_{33} - 4\Delta c_{55}}{\lambda^{(0)} + 2\mu^{(0)}}\sin^2\vartheta_p + \frac{1}{4}\frac{\Delta c_{11}}{\lambda^{(0)} + 2\mu^{(0)}}\sin^2\vartheta_p\tan^2\vartheta_p\right], \quad (30)$$

$$R_{qSVqSV} = \varepsilon i\left[-\frac{1}{4}\left(\frac{\Delta c_{55}}{\mu^{(0)}} + \frac{\Delta\rho}{\rho^{(0)}}\right) + \frac{1}{4}\frac{\Delta\rho}{\rho^{(0)}}\tan^2\vartheta_s + \frac{1}{4}\frac{\Delta c_{11} - 2\Delta c_{13} + \Delta c_{33} + 3\Delta c_{55}}{\mu^{(0)}}\sin^2\vartheta_s - \frac{1}{4}\frac{\Delta c_{55}}{\mu^{(0)}}\sin^2\vartheta_s\tan^2\vartheta_s\right], \quad (31)$$

$$R_{SHSH} = \varepsilon i\left[-\frac{1}{4}\left(\frac{\Delta c_{44}}{\mu^{(0)}} + \frac{\Delta\rho}{\rho^{(0)}}\right) + \frac{1}{4}\left(\frac{\Delta c_{66}}{\mu^{(0)}} - \frac{\Delta\rho}{\rho^{(0)}}\right)\tan^2\vartheta_s\right]. \quad (32)$$

Now we specify our result for the case of two isotropic half-spaces. We then have

$$\Delta c_{11} = \Delta c_{33}, \qquad \Delta c_{44} = \Delta c_{55} = \Delta c_{66}, \qquad \Delta c_{13} = \Delta c_{11} - 2\Delta c_{44}. \quad (33)$$

We linearize

$$\Delta c_{11} = \Delta(\rho v_p^2) \approx 2\rho^{(0)}v_p^{(0)}\Delta v_p + v_p^{(0)2}\Delta\rho, \quad (34)$$
$$\Delta c_{44} = \Delta(\rho v_s^2) \approx 2\rho^{(0)}v_s^{(0)}\Delta v_s + v_s^{(0)2}\Delta\rho,$$

$$v_p^{(0)} = \sqrt{\frac{\lambda^{(0)} + 2\mu^{(0)}}{\rho^{(0)}}}, \qquad v_s^{(0)} = \sqrt{\frac{\mu^{(0)}}{\rho^{(0)}}}. \quad (35)$$

With the choice of the averages of v_p, v_s and ρ of both half-spaces as a reference medium in eqs (7)–(9), we obtain the known results for the reflection coefficients for two isotropic half-spaces (Aki & Richards 1980).

Next we specify our results for the case of two transversely isotropic half-spaces. The result of Thomsen (1993) for the R_{qPqP} coefficient with the correction by Rüger (1995) in a linearized form is given by

$$R_{qPqP} = \frac{1}{2}\frac{\Delta(\rho v_{p0})}{\overline{\rho v_{p0}}} + \frac{1}{2}\left[\frac{\Delta v_{p0}}{\bar{v}_{p0}} - \left(2\frac{\bar{v}_{s0}}{\bar{v}_{p0}}\right)^2\frac{\Delta(\rho v_{s0}^2)}{\overline{\rho v_{s0}^2}} + \Delta\left(\frac{c_{13} - c_{33} + 2c_{55}}{c_{33}}\right)\right]\sin^2\vartheta_p + \frac{1}{2}\left[\frac{\Delta v_{p0}}{\bar{v}_{p0}} + \Delta\left(\frac{c_{11} - c_{33}}{2c_{33}}\right)\right]\sin^2\vartheta_p\tan^2\vartheta_p. \quad (36)$$

Here the lower index 0 denotes vertical incidence and the overbar denotes the average of the quantities of both half-spaces.

We write the following approximation of eq. (36):

$$R_{qPqP} \approx \frac{1}{2}\left(\frac{\Delta v_{p0}}{\bar{v}_{p0}} + \frac{\Delta\rho}{\bar{\rho}}\right) + \frac{1}{2}\left(\frac{\Delta v_{p0}}{\bar{v}_{p0}} + \frac{\Delta c_{13} - \Delta c_{33} - 2\Delta c_{55}}{\overline{\rho v_{p0}^2}}\right)\sin^2\vartheta_p + \frac{1}{4}\left(\frac{\Delta c_{11}}{\overline{\rho v_{p0}^2}} - \frac{\Delta\rho}{\bar{\rho}}\right)\sin^2\vartheta_p\tan^2\vartheta_p \quad (37)$$
$$= \frac{1}{2}\left(\frac{\Delta v_{p0}}{\bar{v}_{p0}} + \frac{\Delta\rho}{\bar{\rho}}\right) - \frac{1}{4}\frac{\Delta\rho}{\bar{\rho}}\tan^2\vartheta_p + \frac{1}{4}\left(\frac{\Delta\rho}{\bar{\rho}} + 2\frac{\Delta v_{p0}}{\bar{v}_{p0}} + \frac{2\Delta c_{13} - 2\Delta c_{33} - 4\Delta c_{55}}{\overline{\rho v_{p0}^2}}\right)\sin^2\vartheta_p + \frac{1}{4}\frac{\Delta c_{11}}{\overline{\rho v_{p0}^2}}\sin^2\vartheta_p\tan^2\vartheta_p. \quad (38)$$

Using $\lambda^{(0)} + 2\mu^{(0)} = \overline{\rho v_{p0}^2}$ and $\rho^{(0)} = \bar{\rho}$, we can approximate our result (30) by eq. (38).

In the same way, one can show the consistency of our R_{qSVqSV} and R_{SHSH} coefficients (eqs 31 and 32)—specified in the case of two transversely isotropic half-spaces—with the results of Thomsen (1993, eqs A23 and A25). The reflection coefficients (30)–(32) represent a generalization of formulae previously obtained to weak anisotropy of arbitrary form with

Table 1. Differences in elastic moduli (in GPa) and density (in g cm^{-3}) of both half-spaces for five different azimuths of the receiver profile.

φ	Δc_{11}	Δc_{33}	Δc_{44}	Δc_{55}	Δc_{66}	Δc_{13}	$2\Delta c_{13}-\Delta c_{33}-4\Delta c_{55}$	$\Delta c_{11}-2\Delta c_{13}+\Delta c_{33}+3\Delta c_{55}$	$\Delta\rho$
0°	71.2	-3.8	-17.3	-3.0	-5.0	-5.8	4.2	70.0	- 0.276
30°	23.2	-3.8	-13.7	-6.6	11.5	-6.0	18.2	11.6	- 0.276
45°	-13.8	-3.8	-10.1	-10.1	17.0	-6.3	31.6	- 35.3	- 0.276
60°	-39.8	-3.8	-6.6	-13.7	11.5	-6.5	45.6	- 71.7	- 0.276
90°	-54.8	-3.8	-3.0	-17.3	-5.0	-6.8	59.4	- 96.9	- 0.276

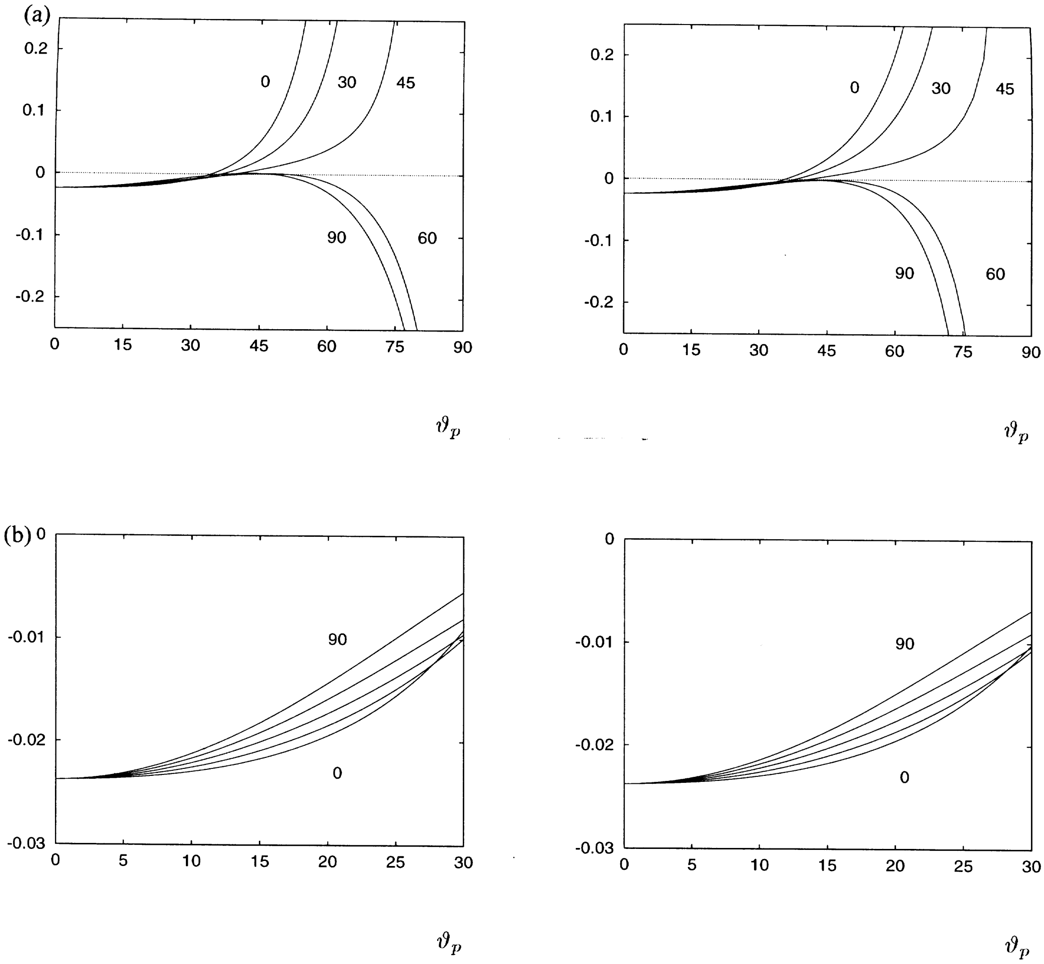

Figure 1. (a) Imaginary part of the R_{PP} reflection coefficient. Left: exact numerical solution; right: approximation by eq. (30). (b) Detail of (a).

the restriction that the qS waves propagate in a symmetry plane of half-space [1]*.

EXAMPLES

A criterion for the range of applicability of eqs (30)–(32) follows from perturbation theory (Paper I). We use eqs (30)–(32) to explain qualitatively the behaviour of the reflection coefficient curves. This is demonstrated in the following by a comparison with numerically calculated reflection coefficients.

We reproduce an example for reflection between the Earth's crust and the Earth's mantle from the paper by Keith & Crampin (1977). Half-space [1] is isotropic with P-wave velocity $v_p^{[1]} = 8.38$ km s^{-1}, S-wave velocity $v_s^{[1]} = 4.83$ km s^{-1} and density $\rho^{[1]} = 3.6$ g cm^{-3}. Half-space [2] represents the (001) cut of orthorhombic olivine with density $\rho^{[2]} = 3.324$ g cm^{-3}. As a reference medium (eqs 7–9) we use $\lambda^{(0)} + 2\mu^{(0)} = 248.2$ GPa, $\mu^{(0)} = 83.2$ GPa, $\rho^{(0)} = 3.462$ g cm^{-3}, which represents an average of the Voigt averages of the half-spaces (e.g. Fedorov 1968). The differences in elastic moduli and in density are

* During the revision of our manuscript we received a manuscript from Vávryčuk & Pšenčík (1997) who by similar techniques derived a formula for the R_{qPqP} coefficient only.

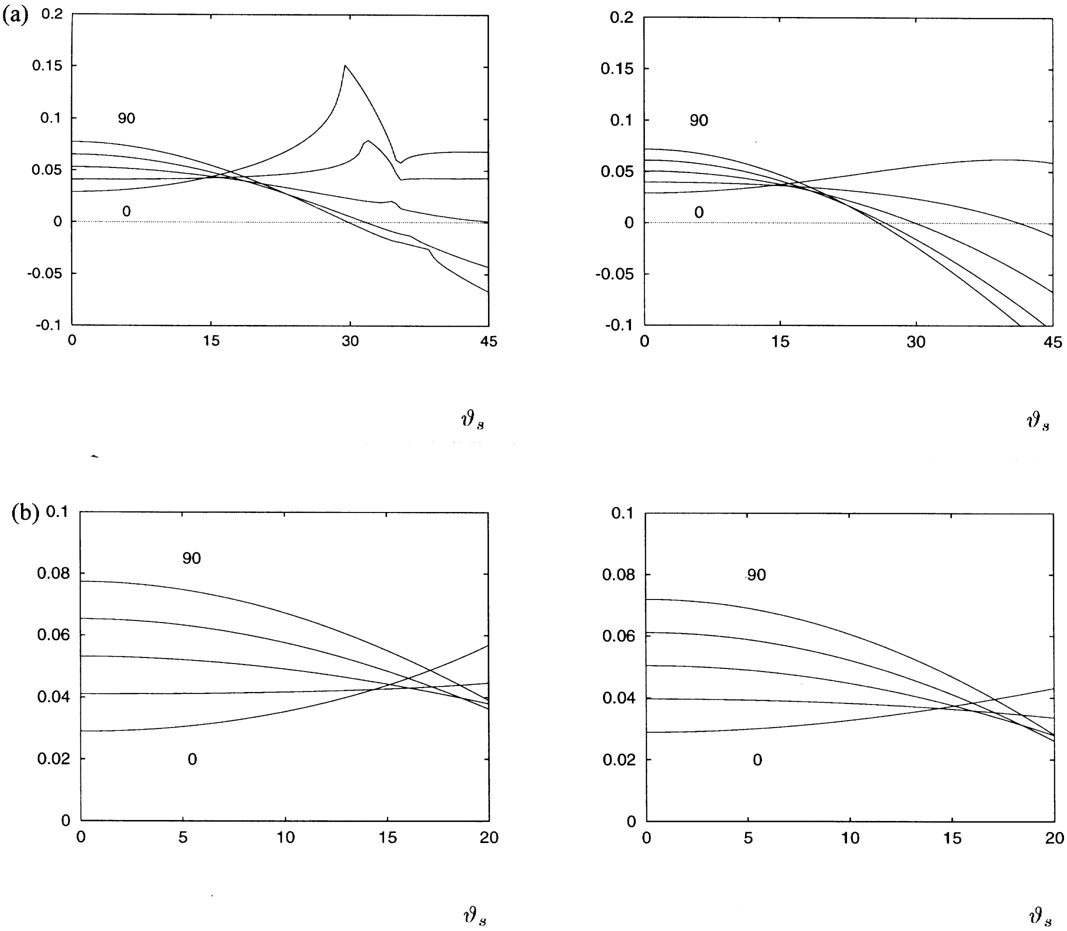

Figure 2. Imaginary part of the R_{SVSV} reflection coefficient. Left: exact numerical solution; right: approximation by eq (31). (b) Detail of (a).

given in Table 1 for five profiles with different azimuthal angles $\varphi = 0°$, $30°$, $45°$, $60°$, $90°$. The left-hand parts of Figs 1–3 show the imaginary part of the numerically calculated exact coefficients as a function of the incident angles ϑ_p and ϑ_s. The right-hand parts show the approximation according to eqs (30)–(32).

We are able to explain all features of the R_{PP} reflection coefficient curves (Fig. 1a) with formula (30). In this specific example the vertical-incidence reflection coefficient is dominated by the term $\sim \Delta\rho$ (see Table 1 and eq. 30). The behaviour of the curves for incidence angles greater than $\vartheta_p = 45°$ is determined by the values of Δc_{11} together with the term $\sim \Delta\rho$. The splitting of the curves in the near-offset range (Fig. 1b) is determined by the values of $2\Delta c_{13} - \Delta c_{33} - 4\Delta c_{55}$ and the common positive gradient of all curves by $\Delta\rho$.

In Fig. 2(a) we show the R_{SVSV} coefficient. The influence of the critical angles of the qP waves near $\vartheta_s = 30°$ cannot be described using our formula. Eqs (30)–(32) always give purely imaginary reflection coefficients, whereas the exact coefficients become complex when one of the waves is an inhomogeneous wave. Fig. 2(b) shows the undercritical range in more detail. The vertical incidence is dominated by the term $\sim \Delta\rho$ and the splitting of the curves comes from the different values of $\Delta c_{11} - 2\Delta c_{13} + \Delta c_{33} + 3\Delta c_{55}$.

In Fig. 3(a) we show the R_{SHSH} coefficient. The profiles with azimuths $\varphi = 0°$ and $\varphi = 90°$ are symmetry planes of the orthorhombic lower half-space. For these profiles there is a decoupling of the SH waves from the qP and qSV waves and we can use eq. (32) for incidence angles up to the critical angle of the transmitted SH wave near $\vartheta_s = 85°$, which is indicated by a steep decrease of the imaginary part of the coefficient. For the other three profiles we have a critical angle of the qP waves near $\vartheta_s = 30°$. In this case we can use eq. (32) only for the near-offset range (Fig. 3b). Here we see the influence of the critical angles of the qP waves near $\vartheta_s = 30°$.

CONCLUSIONS

Approximate R_{qPqP}, R_{qSVqSV} and R_{SHSH} reflection coefficients are derived for the case of a weak-contrast interface between two weakly anisotropic half-spaces in welded contact. They represent a generalization of formulae previously obtained which were restricted to elastic media with a high degree of symmetry.

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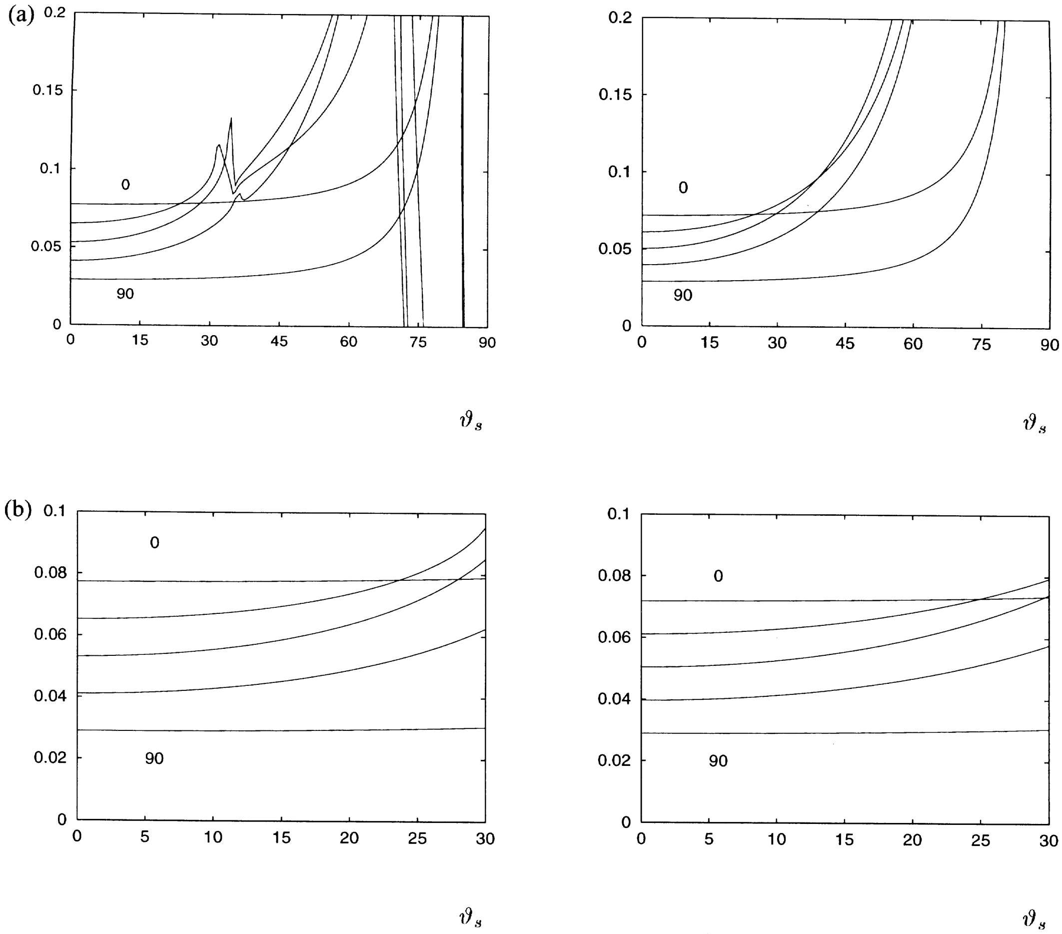

Figure 3. (a) Imaginary part of the R_{SHSH} reflection coefficient. Left: exact numerical solution; right: approximation by eq (32). (b) Detail of (a).

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APPENDIX A: THE MATRICES S, F AND H

We changed the sign of the third vector of the basis $\mathbf{X}^{(0)}$ (Paper I, eqs A3 and A5) to be in accordance with the sign convention of Aki & Richards (1980) for the polarization vectors in the case of two isotropic half-spaces. As a consequence, the corresponding elements of $\mathbf{S}$ have a change in sign.

The elements of the matrix **S** are given by

$$S_{11}=-\frac{1}{2q_P\rho^{(0)}}\left[p^4c_{11}^{(1)}+2p^2q_P^2c_{13}^{(1)}+4p^3q_Pc_{15}^{(1)}+q_P^4c_{33}^{(1)}+4pq_P^3c_{35}^{(1)}+4p^2q_P^2c_{55}^{(1)}-\frac{\rho^{(0)}}{\lambda^{(0)}+2\mu^{(0)}}\rho^{(1)}\right], \tag{A1}$$

$$S_{12}=-\frac{1}{2q_P\rho^{(0)}}\sqrt{\frac{\rho^{(0)}q_P}{\mu^{(0)}q_S}}[p^2q_Sc_{14}^{(1)}+p^3c_{16}^{(1)}+q_P^2q_Sc_{34}^{(1)}+pq_P^2c_{36}^{(1)}+2pq_Pq_Sc_{45}^{(1)}+2p^2q_Pc_{56}^{(1)}],$$

$$S_{13}=-\frac{1}{2q_P\rho^{(0)}}\sqrt{\frac{q_P}{q_S}}[p^3q_Sc_{11}^{(1)}+pq_S(q_P^2-p^2)c_{13}^{(1)}+p^2(q_S^2-p^2+2q_Pq_S)c_{15}^{(1)}-pq_P^2q_Sc_{33}^{(1)}-(2p^2q_Pq_S-q_P^2(q_S^2-p^2))c_{35}^{(1)}$$
$$+2pq_P(q_S^2-p^2)c_{55}^{(1)}-p(q_S-q_P)\rho^{(1)}],$$

$$S_{14}=-\frac{1}{2q_P\rho^{(0)}}i[p^4c_{11}^{(1)}+2p^2q_P^2c_{13}^{(1)}+q_P^4c_{33}^{(1)}-4p^2q_P^2c_{55}^{(1)}+(q_P^2-p^2)\rho^{(1)}],$$

$$S_{15}=-\frac{1}{2q_P\rho^{(0)}}i\sqrt{\frac{\rho^{(0)}q_P}{\mu^{(0)}q_S}}[-p^2q_Sc_{14}^{(1)}+p^3c_{16}^{(1)}-q_P^2q_Sc_{34}^{(1)}+pq_P^2c_{36}^{(1)}-2pq_Pq_Sc_{45}^{(1)}+2p^2q_Pc_{56}^{(1)}],$$

$$S_{16}=-\frac{1}{2q_P\rho^{(0)}}i\sqrt{\frac{q_P}{q_S}}[p^3q_Sc_{11}^{(1)}+pq_S(q_P^2-p^2)c_{13}^{(1)}-p^2(q_S^2-p^2-2q_Pq_S)c_{15}^{(1)}-pq_P^2q_Sc_{33}^{(1)}-(2p^2q_Pq_S+q_P^2(q_S^2-p^2))c_{35}^{(1)}$$
$$-2pq_P(q_S^2-p^2)c_{55}^{(1)}-p(q_P+q_S)\rho^{(1)}],$$

$$S_{22}=-\frac{1}{2q_S\mu^{(0)}}[q_S^2c_{44}^{(1)}+2pq_Sc_{46}^{(1)}+p^2c_{66}^{(1)}-\rho^{(1)}],$$

$$S_{23}=-\frac{1}{2q_S\mu^{(0)}}\sqrt{\frac{\mu^{(0)}}{\rho^{(0)}}}[pq_S^2c_{14}^{(1)}+p^2q_Sc_{16}^{(1)}-pq_S^2c_{34}^{(1)}-p^2q_Sc_{36}^{(1)}+q_S(q_S^2-p^2)c_{45}^{(1)}+p(q_S^2-p^2)c_{56}^{(1)}],$$

$$S_{24}=-\frac{1}{2q_S\mu^{(0)}}i\sqrt{\frac{\mu^{(0)}q_S}{\rho^{(0)}q_P}}[p^2q_Sc_{14}^{(1)}+p^3c_{16}^{(1)}+q_P^2q_Sc_{34}^{(1)}+pq_P^2c_{36}^{(1)}-2pq_Pq_Sc_{45}^{(1)}-2p^2q_Pc_{56}^{(1)}],$$

$$S_{25}=-\frac{1}{2q_S\mu^{(0)}}i[-q_S^2c_{44}^{(1)}+p^2c_{66}^{(1)}-\rho^{(1)}],$$

$$S_{26}=-\frac{1}{2q_S\mu^{(0)}}i\sqrt{\frac{\mu^{(0)}}{\rho^{(0)}}}[pq_S^2c_{14}^{(1)}+p^2q_Sc_{16}^{(1)}-pq_S^2c_{34}^{(1)}-p^2q_Sc_{36}^{(1)}-q_S(q_S^2-p^2)c_{45}^{(1)}-p(q_S^2-p^2)c_{56}^{(1)}],$$

$$S_{33}=-\frac{1}{2q_S\rho^{(0)}}\left[p^2q_S^2c_{11}^{(1)}-2p^2q_S^2c_{13}^{(1)}+2pq_S(q_S^2-p^2)c_{15}^{(1)}+p^2q_S^2c_{53}^{(1)}-2pq_S(q_S^2-p^2)c_{35}^{(1)}+(q_S^2-p^2)^2c_{55}^{(1)}-\frac{\rho^{(0)}}{\mu^{(0)}}\rho^{(1)}\right],$$

$$S_{34}=-\frac{1}{2q_S\rho^{(0)}}i\sqrt{\frac{q_S}{q_P}}[p^3q_Sc_{11}^{(1)}+pq_S(q_P^2-p^2)c_{13}^{(1)}+p^2(q_S^2-p^2-2q_Pq_S)c_{15}^{(1)}-pq_P^2q_Sc_{33}^{(1)}+(q_P^2(q_S^2-p^2)+2p^2q_Pq_S)c_{35}^{(1)}$$
$$-2pq_P(q_S^2-p^2)c_{55}^{(1)}-p(q_P+q_S)\rho^{(1)}],$$

$$S_{35}=-\frac{1}{2q_S\rho^{(0)}}i\sqrt{\frac{\rho^{(0)}}{\mu^{(0)}}}[-pq_S^2c_{14}^{(1)}+p^2q_Sc_{16}^{(1)}+pq_S^2c_{34}^{(1)}-p^2q_Sc_{36}^{(1)}-q_S(q_S^2-p^2)c_{45}^{(1)}+p(q_S^2-p^2)c_{56}^{(1)}],$$

$$S_{36}=-\frac{1}{2q_S\rho^{(0)}}i[p^2q_S^2c_{11}^{(1)}-2p^2q_S^2c_{13}^{(1)}+p^2q_S^2c_{33}^{(1)}-(q_S^2-p^2)^2c_{55}^{(1)}-(q_S^2-p^2)\rho^{(1)}],$$

$$S_{44}=-\frac{1}{2q_P\rho^{(0)}}\left[-p^4c_{11}^{(1)}-2p^2q_P^2c_{13}^{(1)}+4p^3q_Pc_{15}^{(1)}-q_P^4c_{33}^{(1)}+4pq_P^3c_{35}^{(1)}-4p^2q_P^2c_{55}^{(1)}+\frac{\rho^{(0)}}{\lambda^{(0)}+2\mu^{(0)}}\rho^{(1)}\right],$$

$$S_{45}=-\frac{1}{2q_P\rho^{(0)}}\sqrt{\frac{\rho^{(0)}q_P}{\mu^{(0)}q_S}}[p^2q_Sc_{14}^{(1)}-p^3c_{16}^{(1)}+q_P^2q_Sc_{34}^{(1)}-pq_P^2c_{36}^{(1)}-2pq_Pq_Sc_{45}^{(1)}+2p^2q_Pc_{56}^{(1)}],$$

$$S_{46}=-\frac{1}{2q_P\rho^{(0)}}\sqrt{\frac{q_P}{q_S}}\left[-p^3q_Sc_{11}^{(1)}-pq_S(q_P^2-p^2)c_{13}^{(1)}+p^2(q_S^2-p^2+2q_Pq_S)c_{15}^{(1)}+pq_P^2q_Sc_{33}^{(1)}+(q_P^2(q_S^2-p^2)-2p^2q_Pq_S)c_{35}^{(1)}\right.$$
$$\left.-2pq_P(q_S^2-p^2)c_{55}^{(1)}+p(q_S-q_P)\rho^{(1)}\right],$$

$$S_{55}=-\frac{1}{2q_S\mu^{(0)}}\left[-q_S^2c_{44}^{(1)}+2pq_Sc_{46}^{(1)}-p^2c_{66}^{(1)}+\rho^{(1)}\right],$$

$$S_{56}=-\frac{1}{2q_S\mu^{(0)}}\sqrt{\frac{\mu^{(0)}}{\rho^{(0)}}}\left[pq_S^2c_{14}^{(1)}-p^2q_Sc_{16}^{(1)}-pq_S^2c_{34}^{(1)}+p^2q_Sc_{36}^{(1)}-q_S(q_S^2-p^2)c_{45}^{(1)}+p(q_S^2-p^2)c_{56}^{(1)}\right],$$

$$S_{66}=-\frac{1}{2q_S\rho^{(0)}}\left[-p^2q_S^2c_{11}^{(1)}+2p^2q_S^2c_{13}^{(1)}+2pq_S(q_S^2-p^2)c_{15}^{(1)}-p^2q_S^2c_{33}^{(1)}-2pq_S(q_S^2-p^2)c_{35}^{(1)}-(q_S^2-p^2)^2c_{55}^{(1)}+\frac{\rho^{(0)}}{\mu^{(0)}}\rho^{(1)}\right];$$

$$q_P=\sqrt{\frac{\rho^{(0)}}{\lambda^{(0)}+2\mu^{(0)}}-p^2},\qquad \sqrt{q_p}>0:\ p^2<\frac{\rho^{(0)}}{\lambda^{(0)}+2\mu^{(0)}};$$

$$q_P=i\sqrt{p^2-\frac{\rho^{(0)}}{\lambda^{(0)}+2\mu^{(0)}}},\qquad \mathcal{I}m\sqrt{q_P}>0:\ p^2>\frac{\rho^{(0)}}{\lambda^{(0)}+2\mu^{(0)}};$$

$$q_S=\sqrt{\frac{\rho^{(0)}}{\mu^{(0)}}-p^2},\qquad \sqrt{q_S}>0:\ p^2<\frac{\rho^{(0)}}{\mu^{(0)}};\qquad q_S=i\sqrt{p^2-\frac{\rho^{(0)}}{\mu^{(0)}}},\qquad \mathcal{I}m\sqrt{q_S}>0:\ p^2>\frac{\rho^{(0)}}{\mu^{(0)}}.$$

The matrix for rotation of the isotropic vectors in zero-order perturbation theory is

$$\mathbf{F}=\begin{pmatrix}1&0&0&0&0&0\\0&f_{22}^{(0)}&-f_{23}^{(0)}&0&0&0\\0&f_{23}^{(0)}&f_{22}^{(0)}&0&0&0\\0&0&0&1&0&0\\0&0&0&0&f_{55}^{(0)}&-f_{56}^{(0)}\\0&0&0&0&f_{56}^{(0)}&f_{55}^{(0)}\end{pmatrix},\tag{A2}$$

$$f_{22}^{(0)}=\frac{S_{33}-q_2^{(1)}}{\sqrt{(S_{33}-q_2^{(1)})^2+S_{23}^2}},\qquad f_{23}^{(0)}=\frac{-S_{23}}{\sqrt{(S_{33}-q_2^{(1)})^2+S_{23}^2}},\qquad q_{2,3}^{(1)}=\frac{1}{2}\left[S_{22}+S_{33}\mp\sqrt{(S_{22}-S_{33})^2+4S_{23}^2}\right],$$

$$f_{55}^{(0)}=\frac{S_{66}-q_5^{(1)}}{\sqrt{(S_{66}-q_5^{(1)})^2+S_{56}^2}},\qquad f_{56}^{(0)}=\frac{-S_{56}}{\sqrt{(S_{66}-q_5^{(1)})^2+S_{56}^2}},\qquad q_{5,6}^{(1)}=\frac{1}{2}\left[S_{55}+S_{66}\pm\sqrt{(S_{55}-S_{66})^2+4S_{56}^2}\right],$$

$$\mathbf{F}^{-1}=\mathbf{F}^{\mathrm{T}},\tag{A3}$$

$\mathcal{I}m\geq 0$ for all square roots in eq. (A2). The signs in front of the square roots in the equations for $q_{2,3}^{(1)}$ and $q_{5,6}^{(1)}$ are chosen in such a way that $q_2^{(1)}<q_3^{(1)}$ and $q_5^{(1)}>q_6^{(1)}$ for real q_S to achieve the following sequence of vectors with qS_1 the faster and qS_2 the slower of the qS waves: (qP^{U}, qS_1^{U}, qS_2^{U}, qP^{D}, qS_1^{D}, qS_2^{D}), where the indices U denote upgoing and D downgoing waves.

The first-order linear factors for the perturbation of the isotropic stress–displacement vectors are

$$\mathbf{H}=\begin{pmatrix}0&h_{21}^{(1)}&h_{31}^{(1)}&h_{41}^{(1)}&h_{51}^{(1)}&h_{61}^{(1)}\\h_{12}^{(1)}&0&h_{32}^{(1)}&h_{42}^{(1)}&h_{52}^{(1)}&h_{62}^{(1)}\\h_{13}^{(1)}&h_{23}^{(1)}&0&h_{43}^{(1)}&h_{53}^{(1)}&h_{63}^{(1)}\\h_{14}^{(1)}&h_{24}^{(1)}&h_{34}^{(1)}&0&h_{54}^{(1)}&h_{64}^{(1)}\\h_{15}^{(1)}&h_{25}^{(1)}&h_{35}^{(1)}&h_{45}^{(1)}&0&h_{65}^{(1)}\\h_{16}^{(1)}&h_{26}^{(1)}&h_{36}^{(1)}&h_{46}^{(1)}&h_{56}^{(1)}&0\end{pmatrix},\tag{A4}$$

$$h_{mn}^{(1)}=\frac{V_{nm}}{q_m^{(0)}-q_n^{(0)}},\qquad m,n=1\ldots6,\qquad m\neq n,\qquad \{mn\}\bar{\in}\{23,32,56,65\},$$

$$h_{23}^{(1)}=-h_{32}^{(1)}=\frac{V_{31}h_{21}^{(1)}+V_{34}h_{24}^{(1)}+V_{35}h_{25}^{(1)}+V_{36}h_{26}^{(1)}}{V_{22}-V_{33}},\qquad h_{56}^{(1)}=-h_{65}^{(1)}=\frac{V_{61}h_{51}^{(1)}+V_{62}h_{52}^{(1)}+V_{63}h_{53}^{(1)}+V_{64}h_{54}^{(1)}}{V_{55}-V_{66}}.$$