A 2.5D EMHD ANALYTICAL MODEL
OF STEADY-STATE HALL MAGNETIC RECONNECTION
IN A COLLISIONLESS COMPRESSIBLE PLASMA

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Abstract. A 2.5D analytical EMHD model of steady-state magnetic reconnection in a collisionless compressible plasma with a constant electron temperature is developed. It is shown that as in incompressible case, the solution of the Grad-Shafranov equation for the magnetic potential is a basis for the problem analysis. The formation of the double electric layers and layers of low density plasma, mapping the magnetic separatrices, are investigated. It is found that formation of depletion layers should not be governed by the out-of-plane magnetic field, but rather, the origin of these layers lies inside the electron diffusion region, where the number density should attain the local minimum due to the inner dissipative processes. The double electric layers are found to be thin separatrices-elongated sheets whose cross-sections are of the order of the electron diffusion region half-width. These charged layers provide the presence of the strong electric field orthogonal to the in-plane magnetic field, which forces electrons to accelerate in the out-of-plane direction. Outside of the double electric layers, the condition of quasi-neutrality of plasma is found to be fulfilled to high accuracy. The extreme values of the electron velocity and electric field are found to be controlled by the width of the electron diffusion region. Thus, last one seems to be the fundamental parameter of the reconnection process.

The problem formulation
The problem formulation and the technique of solving are very similar to those used in our work dedicated to the case of incompressible plasma [Korovinskiy et al., 2008]. We examine the problem of steady-state collisionless magnetic reconnection in a compressible plasma in the vicinity of the \textit{X}-line, at a length scale of the proton inertial length, $l_p = c(m_p/4\pi n_p e^2)^{1/2}$, where $m_p$ is the proton mass, $n_p$ is the proton number density, $e$ is the elementary charge value, and $c$ is the speed of light. The current sheet is supposed to be infinite, thus, the problem turns out to be 2.5-dimensional. Outside the electron diffusion region, EDR, which is supposed to be small (of the order of the electron inertial length, $l_e = c(m_e/4\pi n_e e^2)^{1/2}$ in its cross-section and 10’s $l_e$ along it), the plasma is assumed to be non-resistive. It is also assumed to be quasi-neutral at the upper boundary of the examined region. We use also a simplifying assumption of the homogeneity of the electron temperature. Our coordinate system is chosen as follows: The \textit{Z}-axis is normal to the current sheet, the \textit{Y}-axis is directed along the \textit{X}-line, and the \textit{X}-axis coincides with the proton outflow direction. By using normalized variables, the problem is described by the EMHD equations augmented by the equation for the ion velocity,

\begin{align}
(1) & \quad (V_p \cdot \nabla V_p) = -\left(1/n_p\right)\nabla P_p + E + V_p \times B, \\
(2) & \quad E + V_e \times B = -\left(1/n_e\right)\nabla P_e, \\
(3, 4) & \quad \nabla \times B = -n_e V_e, \quad \nabla \times E = 0, \\
(5, 6) & \quad \nabla \cdot B = 0, \quad \nabla \cdot E = \left(c^2/V_d^2\right)\cdot(n_p - n_e), \\
(7) & \quad \nabla \cdot \left(n_{p,e} V_{p,e}\right) = 0, \\
(8, 9) & \quad P_{p,e} = n_{p,e} T_{p,e}, \quad T_e = \text{const.}
\end{align}
Here, \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields, respectively, \( \mathbf{V} \) is the bulk velocity, \( P \) is the scalar pressure, and \( T \) is the temperature. Subscripts \( (p,e) \) mean “proton” and “electron”, respectively. The set of the constants of normalization contains the magnetic field value and number density at the upper boundary of the examined region in the center of the sheet, \( B_0 = B_0(0; z_{max}) \) and \( n_0 = n_{p,e}(0; z_{max}) \). We also use the corresponding proton Alfvén velocity \( V_A = B_0/(4\pi n_0 m_p)^{1/2} \), the Alfvén electric field \( E_A = (1/c)B_0V_A \), the gas pressure \( P_0 = B_0^2/4\pi n_0 \), the temperature \( T_0 = B_0^2/4\pi n_0 \), and the length scale \( l_p \). Note that in our steady-state 2.5D case, the electric field \( E_y \) must be a constant, according to Faraday’s law (4). So, we define \( E_y = \varepsilon \); where \( \varepsilon \) is the reconnect rate which we assume to be small, \( \varepsilon << 1 \).

Electric potential and number density
We introduce the electric potential \( \varphi \) and the effective potential \( \varphi^* \) as follows,

\[
(10,11) \quad \mathbf{E}_\perp = -\nabla \varphi, \quad \varphi^* = \varphi - T_e \cdot \ln(n_e),
\]

From now on, we use the special designation \( \perp \) for the in-plane projections of all vectors. Then, we rewrite Ohm's law (2) as follows, \( [\mathbf{V}_e \times \mathbf{B}] = \text{grad}(\varphi^*) - \mathbf{e}_y \), where \( \mathbf{e}_y \) is the unit vector of the \( Y \)-axis. After multiplying with \( n_e \), the \( Y \)-component of this equation can be written in several equivalent forms,

\[
(12) \quad \varepsilon n_e = n_e V_e B_z - n_e V_e B_y = \frac{\partial B_y}{\partial z} \frac{\partial A}{\partial x} - \frac{\partial B_y}{\partial x} \frac{\partial A}{\partial z} = \left( \frac{\partial (A,B_y)}{\partial (x,z)} \right),
\]

where we use the magnetic potential \( A \) defined as follows,

\[
(13) \quad \mathbf{B}_\perp = \left[ \nabla A \times \mathbf{e}_y \right] _\perp,
\]

and equality \( n_e V_e = -\text{grad}(B_y) \times \mathbf{e}_y \), which follows from Ampere's law (3) and means that the magnetic field \( B_y \) plays the role of the electron flux flow-function.

The left part of expression (12) is non-zero, so, we can use quantities \( A \) and \( B_y \), as a couple of independent variables instead of \((x,z)\). This approach allows expressing the in-plane part of the Ohm law in the following form,

\[
(14) \quad \frac{\partial \varphi^*}{\partial A} = \frac{1}{n_e} \Delta A = V_e, \quad (15) \quad \frac{\partial \varphi^*}{\partial B_y} = \frac{1}{n_e} - B_y,
\]

where \( \Delta A \) is the in-plane part of the Laplace operator. Thus, the solution of our problem (except of the problem of the proton motion) reduces to the solution of the two equations (14) and (15).

With accuracy up to the order of \( B_y^2 \), solution of equations (14,15) has the form,

\[
(16) \quad \varphi = G(A) + \gamma B_y^2, \quad (17) \quad n_e = N(A) \exp(v) + (1/2\gamma)[1 - \exp(v)],
\]

where \( G \) is an unknown function of the magnetic potential, \( \gamma \) is a small parameter, \( N \) is the number density value at the axes (a boundary condition for our solution), and \( v = \gamma B_y^2/T_e \).

After some mathematics, we obtain solutions for quantities \( G(A) \) and \( V_e(A,B_y) \) as follows,

\[
(18) \quad V_e(A,B_y) = V(A) + \frac{T_e}{2\gamma} \frac{N^*}{n_e(A,B_y)} N(A) - 1/(2\gamma), \quad (19) \quad G(A) = \int_0^A V'(A')dA' + T_e \ln \left| \frac{N(A) - 1/(2\gamma)}{N(0) - 1/(2\gamma)} \right|
\]

Here, \( V(A) \) is the electron velocity value at the axes (a boundary condition for the solution), and \( N^* \) is the derivative \( dN/dA \).

Let us address the issue of the \( \gamma \) parameter. Firstly, \( \gamma \) cannot be negative, otherwise the parallel electric field \( E_y = (B_y \text{grad}(\varphi^*)) \cdot \varphi^* \) would be negative, so, \( E_y \) would point in the wrong direction. For \( \gamma = 0 \), we have the limiting case, \( \varphi = G(A), n_e = N(A) - B_y^2/(2T_e) \), when parallel electric field disappears. For \( \gamma = 1/2 \) we have another limiting case, corresponding to the case of incompressible plasma [Korovinskyy et al., 2008]. So, the range of permissible values of \( \gamma \) is \([0, 1/2]\). Considering the derivative \( \partial n_e/\partial B_y^2 \), one can see that the derivative is always negative for \( 0 < \gamma < 1/(2\times \max(N)) \). Thus, the larger \( B_y \), the smaller is \( n_e \).

On the other hand, we know about the existence of the thin layers of low-density plasma localized in the separatrices' vicinities [e.g., Shay et al., 2001; Øieroset et al., 2001; Mozer et al., 2002; Yang et al., 2006]. These vicinities seem to be the only regions where a relatively strong out-of-plane magnetic field occurs (though, some simulations and observations demonstrate the strong \( B_y \) in the whole outflow region [Phan et
Therefore, one may expect that $B_y$ is responsible for the depletion layer formation. Though, upon a closer view, this conclusion turns out to be not incontrovertible. One can easily see that the maximum impact of the out-of-plane magnetic field on the number density is described by the term $B_y^2/(2T_e)$, corresponding to the case of $\gamma = 0$. Since $N(A)$ is a quantity of the order of 1, it is clear that, except of the case of a very cold plasma, the number density demonstrates a relatively weak dependence on $B_y$. In numerical simulations, on the contrary, decrease of the number density in the separatrices' vicinities turns out to be substantial, up to several times [e.g., Shay et al., 2001; Yang et al., 2006]. This means that the distribution $N(A)$ may be responsible for the formation of the depletion layers rather than the out-of-plane magnetic field. In favor of this statement is the fact that regions of a minimum of density and an extreme of the out-of-plane magnetic field are slightly separated spatially. Namely, the depletion layers are localized at the separatrices exactly, while the extremes of $B_y$ are shifted a bit into the outflow regions (see Figs. 2, 5 and 9 in Yang et al., 2006).

To provide the minimum of $n_e$ at the separatrices, $N(A)$ should have a local minimum at the origin, where the separatrices intersect each other. What reason may cause the formation of such minimum? Since the EMHD approximation is inapplicable inside the EDR, we cannot apply any EMHD formula to describe $N(A)$ there, but we can apply general physical arguments. Inside this region, dissipative processes, evidently, may increase the electron temperature. Under the constancy of the total pressure across the boundary layer (which is the case), an increase of the temperature is accompanied by a decrease of the number density. Thus, quantity $n_e$ may attain a local minimum at the origin. In fact, this minimum has been found in numerical simulations [e.g., Pritchett, 2001; Yang et al., 2006].

**Grad-Shafranov equation and out-of-plane magnetic field**

Let us rewrite equation (14) again, making use of solution (18),

$$\Delta_\perp A = \frac{T_e}{2\gamma} N'(A) + n_e(A, B_y) \cdot V(A).$$

For the case of incompressible plasma, this equation turns out to be the Grad-Shafranov equation [Uzdensky and Kulsrud, 2006; Korovinskiy et al., 2008], but the dependence of $n_e$ on $B_y$ complicates the situation significantly. Nevertheless, we are still able to solve an approximate equation, making use of the smallness of the quantity $v$. Writing down the first two members of the Taylor series of expression (17) for $n_e$, and neglecting the summand $\tilde{v}$, we get $n_{e,0} = N(A)$, yielding a Grad-Shafranov equation for $A$,

$$\Delta_\perp A = \frac{dL(A)}{dA}, \quad \text{where} \quad L(A) = \frac{T_e}{2\gamma} \ln \frac{N(A) - 1/(2\gamma)}{N(0) - 1/(2\gamma)} + \int_0^A N(A')V(A')dA'.$$

Then, we apply the procedure worked out in our former work. Using the boundary layer approximation $\partial/\partial x \ll \partial/\partial z$, so $\Delta_\perp \approx \partial^2/\partial z^2$, we multiply equation (21) by $\partial A/\partial z$ and integrate it,

$$\frac{1}{2} \left[ \frac{\partial A}{\partial z} \right]^2 = L(A) - L(A_0),$$

where we introduce the notation $A_0 = A(x,0)$. This way, we arrive at the solution for the magnetic potential,

$$z(A) = \frac{\pm 1}{\sqrt{2}} \int_0^A \frac{dA'}{\sqrt{L(A') - L(A_0)}},$$

where the sign in front of the integral is positive in the lower semiplane and negative in the upper one. Thus, the solution for the magnetic potential $A$ may be calculated if the boundary conditions, $N(A)$ and $V(A)$, are assigned somehow (e.g., extracted from numerical modeling data, or fixed “manually”). Another boundary condition required is the distribution of the magnetic field $B_z(x,0)$, which we need for the calculation of $A_0$.

Then, we obtain an approximate solution for $B_y$ analogously to the incompressible case by using the method of characteristics [e.g., Sobolev, 1989],

$$B_y(r_\perp) = (-1)^{k+1} \int_{r_0}^{r_\perp} \frac{N dl}{B_\perp} + B_y(r_\perp),$$

where $dl$ is an infinitely small displacement along the in-plane projection of the magnetic field line, $r_0\perp$ marks the initial points of integration lying on the EDR boundary, and $k$ is the quadrant number. The second summand in the right-hand side, which is the $B_y$ value at the EDR boundary, may be neglected due to its smallness [Korovinskiy et al., 2008].
Proton motion

Now, we proceed to the solution of the equation of the proton motion (1). The procedure is quite similar to the one applied to Ohm’s law in the previous sections. We start with the $Y$-component of equation (1),

\begin{equation}
(\mathbf{V}_p \cdot \mathbf{\nabla}) \mathbf{V}_{py} = \varepsilon + \left[ \mathbf{V}_p \times \mathbf{B} \right]_y .
\end{equation}

Introducing the flux-function $\Psi_p$ as follows, $n_p \mathbf{V}_p \cdot \mathbf{e}_p = \nabla \times \psi_p \times \mathbf{e}_y$, we get analogously to equation (12),

\begin{equation}
\left[ \mathbf{V}_p \times \mathbf{B} \right]_y = \frac{1}{n_p} \frac{\partial (A, \Psi_p)}{\partial (x, z)} = \frac{\sigma}{n_p}, \quad \text{where} \quad \sigma \equiv \frac{\partial (A, \Psi_p)}{\partial (x, z)} .
\end{equation}

Then, the solution for $V_{py}$ is obtained analogously to the solution for $B_y$ (24),

\begin{equation}
V_{py}(\mathbf{r}_y) = \int_{\mathbf{r}_0}^{\mathbf{r}_y} \frac{n_p \varepsilon + \sigma}{\nabla \times \psi_p} dl + V_{py}(\mathbf{r}_0) ,
\end{equation}

where $dl$ is an infinitely small displacement along the in-plane projection of the proton stream line.

Now, let us consider the in-plane part of equation (1),

Neglecting the last term on the right-hand side of this equation of motion (physically, this means that we neglect the Lorenz force compared to the electric and pressure forces), and applying the vector analysis identity, we can write the internal product of the equation (28) with vector $\mathbf{V}_p \cdot \mathbf{\nabla}$ as follows,

\begin{equation}
\frac{\partial}{\partial A} \left[ \frac{1}{2} V^2_p + \phi \right] = -\frac{1}{n_p} \frac{\partial P_p}{\partial A} ,
\end{equation}

where we apply the equality $\partial/\partial A = -(n_p/\sigma)(\mathbf{V}_p \times \mathbf{B})$, which is obtained by using expression (26). The integral of this equation may be written as follows,

\begin{equation}
\frac{1}{2} V^2_p + \phi + T_p = C(\Psi_p) - I(A, B_y) ,
\end{equation}

where $C(\Psi_p)$ is a constant of integration, and $I(A, B_y) = \int_{-\infty}^{A} \frac{\partial}{\partial A} \ln(n_p) dA$. Now, making use of the boundary layer approximation, we neglect quantity $\partial \Psi_p/\partial x$ compared to quantity $\partial \Psi_p/\partial z$, so, we can write $(n_p \mathbf{V}_p \cdot \mathbf{e}_y)^2 \approx (\partial \Psi_p/\partial z)^2$. With this approximation we derive the solution of the equation (30) for quantity $z(\psi_p)$,

\begin{equation}
z(\Psi_p) = -\text{sign}(V_{px}) \int_{\mathbf{r}_0}^{\mathbf{r}_y} \frac{d\Psi}{\sqrt{2} \sqrt{n_p C(\Psi) - I - \phi - T_p}} .
\end{equation}

To make use of formula (31), we have to calculate firstly quantities $C(\Psi_p)$ and $I(A, B_y)$. Quantity $I$ may be found numerically, when we know $n_p$ and $T_p$. As for the proton concentration, we find it by using equation (6). The proton pressure is computed by formula $(1/2)B^2 + n_e T_e + P_p = \Pi$, where $\Pi$ is the total pressure which is uniform across the boundary layer and assumed to be a fixed function (boundary condition). Knowing $n_p$ and $P_p$ we calculate $T_p$ and quantity $I$. Note that as long as $I$ depends not only on $A$ but also on $B_y$, the inaccuracy of our approach grows, while $|B_y|$ increases. The further steps are quite identical to the procedure presented in our former work. Namely, to calculate $C(\Psi_p)$, we make use of the fact that at the upper boundary of our domain, i. e., at the boundary of EMHD and MHD regions, the frozen-in condition must be fulfilled, so that protons possess a drift velocity there, $V_{p,\text{top}} = [E_{\text{top}} \times \mathbf{B}_{\text{top}}]/(B_{\text{top}})^2$. Knowing the values of all quantities at the upper boundary, we calculate $\Psi_p(\infty)$ and $\sigma(\infty)$,

\begin{equation}
\Psi_p(\infty) = \int_{0}^{\infty} n_p(\infty) V_{p,\text{top}}(x') dx' , \quad \sigma(\infty) = \frac{1}{2} \left( V^2_{p,\text{top}} + \phi_{\text{top}} + T_{p,\text{top}} + I_{\text{top}} \right) ,
\end{equation}

wherefrom we get function $C(\Psi_p)$. Knowing $\Psi_p$, we obtain $V_{py}$, and thus, we get the full solution of the problem.

Summary

In the previous sections we presented a 2.5D EMHD model of steady-state magnetic reconnection in a collisionless compressible plasma, developed under the assumption that $T_e = \text{const}$. This model generalizes the solution obtained for incompressible plasma [Korovinskiy et al., 2008]. All essential features of the
incompressible model stay valid, though the solution attains some new features. As before, the acceleration of electrons in the out-of-plane direction inside the EDR and in the separatrices’ vicinities seems to play the key role in the whole process, due to the tight correlation between the electron velocity \( V_{\gamma} \) and the distribution of the electric field potential. It is noteworthy that the mechanism of this acceleration is not the same in the two regions. Inside the EDR, electrons are demagnetized, so that the electric field \( E_x \) accelerates them freely. In the separatrices’ vicinities, the acceleration of electrons is supported, evidently, by \( E \times B \) drift including the surfing mechanism [Hoshino, 2005]. Nevertheless, these two different scenarios result in the same value of the velocity, \( \max |V_{\gamma}| = V_{\gamma}/\delta \), where \( \delta \) is the EDR half-width measured in electron inertial length units. On the other hand, surfing acceleration requires a very strong electric field. For the electron velocity to be of the order of \( V_{\gamma}/\delta \), the electric field strength should reach the value \((m_p/m_e)^{1/2}E_x/\delta\). The presence of the electric field of such intensity is supported in turn by the formation of thin double electric layers mapping the separatrices, as it is shown in Fig.1c. The width of these layers depends also on \( \delta \). Thus, the EDR cross-size seems to be the fundamental parameter of the reconnection process.

The existence of the layers of rarefied plasma mapping the separatrices is another essential feature of Hall magnetic reconnection. Sometimes, the origin of depletion layers is attributed to the impact of the out-of-plane magnetic field [e.g., Yang et al., 2006]. It is reasoned by the fact that the regions with extreme values of the magnetic field \( B_z \) are localized near the separatrices, as are the depletion layers. Nevertheless, under close examination, these two regions do not coincide spatially. Namely, the extreme values of \( B_z \), in contrast to the depletion layers, do not outline the separatrices, but lie inside the outflow region (see Fig. 1a). This means that the out-of-plane magnetic field may take part in the formation of these dips of density, but the main contribution is made by some other factor. According to the model developed here, the formation of the low density regions takes its origin inside the EDR, where dissipative processes give rise to an increase of the temperature, so the pressure balance condition requires the number density to decrease. This density gap spreads along the in-plane magnetic field lines, including separatrices, being deepened by the increasing field \( B_z \), as it is demonstrated in Fig. 1b. Note that from the mathematical point of view, the in-plane magnetic field lines are the characteristics of the problem, and the zero-order solution is defined by the magnetic potential \( A \), while quantity \( B_z \) defines the first-order correction. Therefore, the distribution of the number density is found to satisfy the same rules as the electric potential, the electric field, and the out-of-plane electron velocity.

One more feature of the obtained solution is the dependence of the second (correctional) term of the electric potential and the number density on the unknown parameter \( \gamma \). Permitted values of \( \gamma \) lie in a tight range \([0; 1/2]\), and the solution is not very sensitive to its precise value. So, to investigate analytically the distribution of the number density, or the magnetic field, or other quantities, one may just fix \( \gamma = 0 \) without much loss of accuracy. The only quantity depending critically on it is the electric field, \( E_{\parallel} \), parallel to the in-plane magnetic field. At the same time, this quantity is very small compared to the total electric field, so, generally it can be neglected without serious consequences. Though, concerning the comparison of our model with numerical simulations or observations, the dependence of \( E_{\parallel} \) on \( \gamma \) is of good service to fix the \( \gamma \) value, when \( E_{\parallel} \) is known. This comparison is object of our future study. Encouraging for this matter is that the comparison of our former model of reconnection in an incompressible plasma with numerical simulations showed good qualitative agreement [Semenov et al., 2009].

Finally, the solution of the equation of proton motion is obtained as well. In the case of incompressible plasma, this equation was reduced to the Bernoulli law [Korovinsky et al., 2008]. When the number density is not a constant, this approach does not work, unfortunately. Nevertheless, the solution of the problem is found to be still available, though not everywhere. Particularly, our method does not work in regions of a strong out-of-plane magnetic field, because the measure of inaccuracy is larger, the larger the \( B_z \) contribution to the number density is. For high electron temperatures, this contribution tends to zero and the region of reliability increases up to the whole EMHD domain. Inside the region of reliability, the solution looks credible, because the quantities obtained behave in an expectable manner. Specifically, the outflow proton velocity is found to be of the order of \( V_{\alpha} \), as one can see in Fig. 1d.

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References


![Figure 1](https://example.com/figure1.png)

Figure 1. Some physical quantities are plotted in the first quadrant. Parameters of the numeric model are essentially the same as in our former work [Korovinskiy et al., 2008]. EDR half-width is equal to one $l_s$, $V(A)$ is a $\delta$-like function with extreme value $V_{es}, B_z(x,0)=(\varepsilon/3)x, \varepsilon=0.2, \gamma=0, II=const, T_e=0.1$, and $c=500V_{es}$. Panel a) Out-of-plane magnetic field by color and electron flow stream lines by solid black curves, magnetic separatrix is shown by white curve; panel b) electron number density by color and magnetic separatrix by white curve; panel c) difference between proton and electron number densities; panel d) proton flow stream lines by white curves and proton outflow velocity $V_{px}$ by color.