DOUBLE GRADIENT INSTABILITY IN A COMPRESSIBLE PLASMA
CURRENT SHEET

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Abstract. A linear MHD instability is investigated of the electric current sheet characterized by a small normal magnetic field component $B_z$ varying along the sheet. The tangential magnetic field component, $B_x$, is modeled by a hyperbolic function describing Harris-like variations of the field across the sheet. This work is an extended numerical study of the so called "double gradient instability" which was analyzed previously in the framework of the simplified analytical approach for an incompressible plasma. For this problem, formulated in 3D domain, the conventional compressible ideal MHD equations are applied. By assuming Fourier harmonics along the electric current, the linearized 3D equations have been reduced to 2D ones. A finite difference numerical scheme is applied to examine the time evolution of small initial perturbations of the equilibrium background. Finally, dispersion curve is obtained for the kink-like mode of the instability. It is shown that this curve demonstrates a quantitative agreement with the previous theoretical results, obtained in the frame of a 1D incompressible model. The dependence of the instability growth rates on the magnetic gradient $\partial B_z/\partial x$ is examined, demonstrating a good agreement with the theoretical predictions. However, the numerical growth rates are somewhat less than the analytical ones by a factor depending, probably, on a ratio of the acoustic and Alfvén speeds. This dependence is a subject of our future study.

Introduction
Flapping oscillations of the magnetotail current sheet have been detected by many spacecraft measurements. Namely, CLUSTER observations in the Earth’s magnetotail current sheet indicated the appearance of wave perturbations propagating along the current sheet perpendicular to the magnetic field lines [Sergeev et al., 2003, 2004; Runov et al., 2005, 2006; Petrukovich et al., 2006]. The CLUSTER observations are in favor of the assumption that the flapping perturbations appear more frequently in the central part of the tail, than near the flanks. In the near-flank tail regions, the flapping waves propagate predominantly from the center to the flanks. These observational results confirm the hypothesis about an internal origin of the flapping motions, due to some nonstationary processes (like magnetic reconnection) localized deep inside the magnetotail. The plasma sheet flapping waves are interpreted as quasi-periodic dynamical structures produced by almost vertical slippage motions of the neighboring magnetic tubes, and this allows one to identify them as kink-like perturbations. A data analysis yields a typical frequency of the flapping waves $\omega_f \sim 0.035$ s$^{-1}$ [Sergeev et al., 2003]. A group speed of the flapping waves, estimated from data analysis, is in the range of a few tens (30–70) km·s$^{-1}$ [Runov et al., 2005]. Spatial amplitudes and wavelengths are in the order of 2–5 Earth’s radii [Petrukovich et al., 2006]. A theoretical model of this phenomenon was proposed by Erkaev et al. [2007, 2008, 2009] within the framework of the incompressible MHD approach. In accordance to this model, MHD flapping modes can exist due to a gradient of the normal magnetic field component along the current sheet. A stable situation for the current sheet is associated with a positive result of the multiplication of the two magnetic gradients, and an unstable (wave growth) condition corresponds to a negative result of the product. The analytical solution obtained demonstrates two possible modes of the instability, which are kink-like and sausage-like modes. The kink-like mode is characterized by the displacement of the current sheet center, and an even function $V_z(z)$, while an odd function $V_x(z)$ is relevant to the sausage-like mode, characterized by
variations of the thickness of the current layer without a displacement of its center [Erkaev et al., 2008]. In this paper, we investigate numerically the kink-like mode of the instability and compare our results to theoretical predictions.

**Qualitative Explanation**

The geometrical situation of the problem and the coordinate system are illustrated in Fig. 1. The equation of motion in the frame of an incompressible ideal MHD for nonstationary plasma has a following form,

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) + \nabla P = \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B},
\]

where \( \rho \) is the plasma density, \( \mathbf{V} \) is the plasma bulk velocity, \( \mathbf{B} \) is the magnetic field, \( P \) is the scalar pressure, and \( \mu_0 \) is the magnetic permeability of vacuum.

Let us consider a plasma element of a unit volume, placed at the center of the current layer. In the equilibrium state, the total pressure gradient compensates the magnetic tension,

\[
\frac{\partial P}{\partial z} = \frac{1}{\mu_0} B_s \frac{\partial B_z}{\partial x}.
\]

A small displacement of this plasma element along the z-direction yields the restoring force \( F_z \), which is the difference of two forces, caused by the magnetic tension and the total pressure gradient [Erkaev et al., 2009],

\[
F_z = -\frac{1}{\mu_0} \delta P \left( \frac{\partial B_z}{\partial z} \right)_{z=0}.
\]

In the case of a positive product of two magnetic gradients, the parameter \( \omega_f \) is real, and it has the meaning of the characteristic frequency of the flapping wave oscillations. In the opposite case of a negative product of the magnetic gradients, the current sheet is unstable. The flapping perturbations can grow up exponentially without propagation, because \( \omega_f \) is pure imaginary. These two cases are characterized by a different behavior of the background total pressure. Namely, the total pressure has a maximum at the center of the current sheet for the unstable situation, and it has a minimum for stable conditions.

**The problem formulation**

We start from the system of the conventional equations of ideal MHD for a nonstationary plasma sheet,

\[
\begin{align*}
(1) & \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \\
(2) & \quad \frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \left[ \rho \mathbf{V} \otimes \mathbf{V} + p \mathbf{I} - \frac{1}{4\pi} \left( \mathbf{B} \otimes \mathbf{B} - \frac{\mathbf{B}^2}{2} \mathbf{I} \right) \right] = 0, \\
(3) & \quad \frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} + \rho e + \frac{\mathbf{B}^2}{8\pi} \right) + \nabla \cdot \mathbf{Q} = 0, \\
(4) & \quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{V} - \mathbf{V} \otimes \mathbf{B}) = 0,
\end{align*}
\]

where \( p \) is the gas pressure, \( e \) is the internal energy of a unit volume, \( \mathbf{I} \) is a unit matrix, and \( \mathbf{Q} \) is the energy flux,

\[
\mathbf{Q} = \rho \mathbf{V} \left( \frac{V^2}{2} + e + \frac{p}{\rho} \right) + \frac{1}{4\pi} \mathbf{B} \times \mathbf{V} \times \mathbf{B}.
\]
Assuming an equilibrium initial state, we apply a perturbation technique to solve these equations. Representing all quantities as follows, 
\[ u = u_0 + u_1 \]
where \( u_0 \) is an initial equilibrium value and \( u_1 \) is a small perturbation, we obtain the linearized system of equations, which can be written in the following form,

\[
\frac{\partial U_1}{\partial t} + \frac{\partial F_{x_1}}{\partial x} + \frac{\partial F_{y_1}}{\partial y} + \frac{\partial F_{z_1}}{\partial z} = 0,
\]

where we make use of the adiabatic approximation, \( p = p_0 (\rho / \rho_0)^{\gamma} \), so the vector of the perturbations \( U_1 \) has 7 components only.

\[
U_1 = (\rho_1, \rho_1 \rho V_{0x} + \rho_0 V_{1x}, \rho_1 \rho V_{0y} + \rho_0 V_{1y}, \rho_1 \rho V_{0z} + \rho_0 V_{1z}, B_{1x}, B_{1y}, B_{1z})^T.
\]

We search for a solution of the system (6) in the form of a plain wave propagating in \( y \)-direction, \( U_1 = \delta U(x,z,t) \exp(iky) \). Then, the system of equations for harmonics \( \delta U \) takes the form,

\[
\frac{\partial (\delta U)}{\partial t} + \frac{\partial F_{x_1}}{\partial x} + \frac{\partial F_{z_1}}{\partial z} = S.
\]

Here,

\[
\delta U = (\delta \rho, \rho_0 \delta V_x, \rho_0 \delta V_y, \rho_0 \delta V_z, \delta B_x, \delta B_y, \delta B_z)^T,
\]

\[
F_{x_1} = V_0 \delta \rho + \rho_0 \delta V_x, \quad F_{x_2} = \rho_0 V_{0x} \delta V_x - 2B_{0x} \delta B_x + \delta \rho + (B_0 \cdot \delta B),
\]

\[
F_{x_3} = \rho_0 V_{0x} \delta V_y - B_{0y} \delta B_y - B_{0z} \delta B_z, \quad F_{x_4} = \rho_0 V_{0x} \delta V_z - B_{0x} \delta B_z - B_{0z} \delta B_x,
\]

\[
F_{x_5} = 0, \quad F_{x_6} = V_0 \delta B_x + B_{0x} \delta V_x - B_{0x} \delta B_x - V_0 \delta B_x,
\]

\[
F_{x_7} = V_0 \delta B_x + B_{0x} \delta V_x - B_{0x} \delta B_x - V_0 \delta B_x - B_{0x} \delta B_x - V_0 \delta B_x,
\]

\[
S_1 = -ik \left( \rho_0 \delta V_x \right), \quad S_2 = -ik \left( \rho_0 V_{0x} \delta V_x - B_{0y} \delta B_x - B_{0z} \delta B_y \right) - s_2,
\]

\[
S_3 = -ik \left( \rho_0 V_{0x} \delta V_y - 2B_{0y} \delta B_y + \delta \rho + (B_0 \cdot \delta B) \right) - s_3,
\]

\[
S_4 = -ik \left( \rho_0 V_{0x} \delta V_z - B_{0y} \delta B_y - B_{0z} \delta B_y \right) - s_4,
\]

\[
S_5 = -ik \left( V_0 \delta B_x + B_{0x} \delta V_x - B_{0x} \delta B_x - V_0 \delta B_x \right), \quad S_6 = 0,
\]

\[
S_7 = -ik \left( V_0 \delta B_x + B_{0x} \delta V_x - B_{0x} \delta B_x \right),
\]

where

\[
s_{1,2,3,4} = \frac{\partial V_{0(x,y,z)}}{\partial x} \left( \rho_0 \delta V_x + V_0 \delta \rho \right) + \frac{\partial V_{0(x,y,z)}}{\partial z} \left( \rho_0 \delta V_z + V_0 \delta \rho \right).
\]

### Numeric scheme and results

We solve the system (8) using an Lax-Friedrichs method [Chu, 1978], which is forward in time, centered in space, one-step scheme with an artificial viscosity. The initial equilibrium state is fixed as follows,

\[
B_{0x} = -\tanh(z), \quad B_{0y} = 0, \quad B_{0z} = a + bx; \quad \text{and} \quad V_0 = 0,
\]

and the gas pressure is chosen to satisfy the equilibrium condition, \( \nabla p = j \times B \),

\[
p = p_0 - \frac{ax + bx^2/2}{\cosh^2(z)} - b \ln \left[ \cosh(z) \right] - \frac{\tan^2(z)}{2} - \left( \frac{abx + b^2x^2}{2} \right).
\]

The constant values are: \( \gamma = 5/3, p_0 = 5, \rho_0 = 1, a = 0.1, \) and \( b = 0.1a \) (in some runs, constants \( a \) and \( b \) take other values, but \( b \) is ever 10 times less than \( a \)). The initial distribution of the pressure is shown in Fig. 3. It
possesses a maximum in the center of the sheet, and the product of the gradients of the magnetic field components is negative, so that the physical conditions of the growth of instability are fulfilled. Our calculations are performed in the rectangular box \([x\times z]=([-10...0] \times [-5...5])\), on the uniform computational grid with size \([101 \times 401]\). The initial perturbation is chosen as an even function \(\delta V_z=\exp(-z^2)\). The Courant number is equal to 0.1, so, the time-step approximate value is 0.02. At last, the boundary conditions used are, 
\[
\delta U \bigg|_{z=z_{\text{max}}-z_{\text{min}}} = 0; \quad \frac{\partial (\delta U)}{\partial x} \bigg|_{x=x_{\text{max}}-x_{\text{min}}} = 0.
\]

Firstly, we solve system (8) for a few fixed values of the wave number \(k\) to obtain the dispersion curve \(\omega(k)\) (although \(\omega\) is a pure imaginary quantity, we omit the unit imaginary number below for our convenience). In Fig. 4, the typical picture of the development of the instability for \(k=2\) is presented. Note that the numeric viscosity of the scheme leads to an artificial reduction of the growth rate \(\omega\). The rate of this reduction may be found from the solution of system (8) for \(k=0\). It turns out to be substantial, fluctuating near the value 0.2, so, the results of the calculations must be augmented by this value.

In Fig. 5, the dispersion curve is shown. Another curve, plotted in this figure, demonstrates the theoretical prediction of Erkaev et al. [2007] for \(\omega(k)\), which has the form (for normalized quantities),
\[
(9) \quad \omega = \omega_f \sqrt{\frac{k}{k+1}}; \quad \omega_f = \sqrt{\frac{1}{\rho} \frac{\partial B_z}{\partial z} \frac{\partial B_z}{\partial x}}
\]

Thus, the numeric value of \(\omega_f (0.07)\) turns out to be approximately \(\sqrt{2}\) times less than the theoretical prediction (0.1).

In the next Figure 6, we demonstrate the dependence of the quantity \(\omega_f\) on the magnetic gradient \(\frac{\partial B_z}{\partial x}\). One can see that two curves, the numeric and the theoretical, are nearly parallel to each other, but numerical values are closer to the theoretical ones, the larger the magnetic gradient is. The difference amounts \(\approx 30\%\) for a small magnetic gradient and \(\approx 10\%\) for a large one. So, it seems that numeric growth rate tends to theoretical prediction for a large magnetic gradient. Unfortunately, we are not able to show it convincingly, because the theoretical analysis assumes the magnetic field component \(B_z\) to be small as compared to \(B_x\) [Erkaev et al., 2008]. On the other hand, amplifying the magnetic gradient of \(B_z\) (parameter \(b\)) we have to amplify the \(B_z\) maximum value (parameter \(a\)) to provide the fixed sign of \(B_z\) in the computational area. Therefore, keeping in view the comparison of the numeric results with the theory, we cannot increase \(B_z\) unrestrictedly.

Conclusions
The dispersion curve is calculated numerically for the kink-like mode of the magnetic double gradient instability. It is shown that this curve demonstrates a quantitative agreement with the previous theoretical result obtained in the frame of 1D incompressible model. The numeric discrepancy between the calculations and theory are characterized by a factor \(\sqrt{2}\). The dependence of the instability growth rates on the magnetic gradient \(\frac{\partial B_z}{\partial x}\) is examined as well. It demonstrates a good qualitative agreement with the theoretical predictions, while the numeric disagreement amounts 10–30\%. One of the possible reasons of this disagreement is difference of the mathematical treatments applied. Namely, the theoretical prediction is obtained in the frame of an incompressible MHD, while the numeric calculations are performed for a compressible medium. The influence of this reason may be established more definitely by an increasing of the ratio of the acoustic and Alfven speeds in numeric calculations, which is subject of our future study. Another possible reason is an imperfection of the numeric scheme applied. Thus, the obtained results should be considered as an encouraging intermediate outcome inspiring further investigations.

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References


